On the geometry of lambda-symmetries, and PDEs reduction

Giuseppe Gaeta*
Dipartimento di Matematica, Universitá di Milano
via Saldini 50, I–20133 Milano (Italy)

Paola Morando†
Dipartimento di Matematica, Politecnico di Torino
Corso Duca degli Abruzzi 24, I–10129 Torino (Italy)

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Summary. We give a geometrical characterization of λ-prolongations of vector fields, and hence of λ-symmetries of ODEs. This allows an extension to the case of PDEs and systems of PDEs; in this context the central object is a horizontal one-form μ, and we speak of μ-prolongations of vector fields and μ-symmetries of PDEs. We show that these are as good as standard symmetries in providing symmetry reduction of PDEs and systems, and explicit invariant solutions.

Introduction

The study of differential equations was the main motivation leading S. Lie to create what is now known as the theory of Lie groups; symmetry methods for differential equations have received an ever increasing attention in the last fifteen years, and by now there is an extremely vast literature devoted to these and/or their applications.

It was recently pointed out by Muriel and Romero [8] that, beside standard symmetries, another class of transformations is equally useful in providing symmetry reduction for scalar ordinary differential equations (ODEs); these were christened $\mathcal{C}^\infty$ *symmetries*, or even λ-*symmetries*, as they depend on a smooth scalar function λ (see also [5, 9] for applications of λ-symmetries). Soon afterwards, Pucci and Saccomandi identified the most general class of transformations sharing the "useful" properties of standard symmetries for what concerns reduction of a scalar ODE [12].

*e-mail: g.gaeta@tiscali.it or giuseppe.gaeta@mat.unimi.it
†e-mail: paola.morando@polito.it

In the present note, we extend the concept of λ-symmetries to the case of partial differential equations (PDEs), and to systems. In order to obtain such an extension, we found it convenient to characterize λ-prolongations in $J^{(n)}M$, where (M, π, B) is the space of dependent and independent variables seen as a bundle over the space B of independent variables, in a geometrical way; once this characterization is obtained, it is promptly extended from the ODEs case of $B = \mathbf{R}$ to the PDEs case $B = \mathbf{R}^p$, and to systems.

In the scalar PDE case the transformations of interest depend on a semibasic one form $\mu = \lambda_i \mathrm{d}x^i$, the functions λ_i being such to satisfy a compatibility condition. The transformations of this class leaving invariant the solution manifold for an equation Δ will be said to be μ-symmetries, or Λ^1-symmetries, of Δ. In the case of systems, the form μ takes value in a Lie algebra.

We will thus be able to obtain a sound definition of μ-prolongations and μ-symmetries of (systems of) PDEs. We will also show that, in analogy with the ODE case, μ-symmetries are as useful as standard symmetries in what concerns the symmetry reduction, and the determination of invariant solutions, of (systems of) PDEs. Our approach will suffer from the same limitations as the standard PDE symmetry reduction method.

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1 Prolongations and contact structure

In this section we fix our notation (mainly following the usual one in the field, see e.g. [10]) and recall some basic definition [4, 10, 13, 15].

Let us consider a space $M = B \times U$ with coordinates $x \in B \simeq \mathbf{R}^p$ and $u \in U \simeq \mathbf{R}^q$; when setting a differential equation in this space, we will think of the x as independent variables, and the u as dependent ones.

Thus, more precisely, M will be the total space of a (trivial) linear bundle (M, π, B) over the base space B, with fiber $\pi^{-1}(x) = U$.

Given a bundle P, we will denote $\Gamma[P]$ the set of sections of this bundle, and by $\mathcal{X}[P]$ the set of vector fields in P.

The bundle M can be *prolonged* to the k-th **jet bundle** $(J^{(k)}M, \pi_k, B)$, with $J^{(0)}M \equiv M$; the total space of the jet bundle is also called the jet space, for short.

The jet space $J^{(k)}M$ is naturally equipped with a canonical **contact structure** $\mathcal{E}$, i.e. the module generated by the set of canonical contact one-forms

$$\vartheta^a_J := \mathrm{d}u^a_J - u^a_{J,m} \mathrm{d}x^m$$

with $a = 1, ..., q$, $|J| = 0, ..., k-1$.

The contact structure in $J^{(k)}M$ defines a field of $(p+q)$-dimensional linear spaces in $J^{(k)}M \subset \mathrm{T}((J^{(k-1)}M)$, the contact distribution, corresponding to the tangent subspace spanned by vector fields $Y \in \mathcal{X}[J^{(k)}M]$ annihilated by the contact forms, i.e. such that $Y \lrcorner \theta = 0$ for any contact form θ. The general form of such vector fields is, as well known, $Y = \sum \xi^i D_i^{(k)} + V$.

Here D_i is the total derivative [4, 10, 13, 15] with respect to x^i, $D_i^{(k)}$ its truncation to the k-th jet space, and V is a generic vector field in $\mathcal{X}[J^{(k)}M]$, vertical for the fibration $\pi_{k,k-1} : J^{(k)}M \to J^{(k-1)}M$ (the latter will not appear if we work with infinite-order prolongations; it will however disappear when we deal with a given differential equations and symmetry vector fields for it). The operator $D_i^{(k)}$ reads

$$D_i^{(k)} \ := (\partial/\partial x^i) \ + \sum_{a=1}^q \sum_{|J|=1}^{k-1} \ u^a_{J,i} (\partial/\partial u^a_J) \ .$$

In the following we will write, for ease of notation, simply D_i instead of $D_i^{(k)}$.

A vector field $X \in \mathcal{X}[M]$ can be written, in the (x,u) coordinates, as

$$X \ = \ \xi^i(x,u) \frac{\partial}{\partial x^i} \ + \ \varphi^a(x,u) \frac{\partial}{\partial u^a} \ .$$

This can be uniquely prolonged to a vector field $X^{(k)}$ in $J^{(k)}M$ by requiring it *preserves the contact structure* (the precise meaning of this will be defined in a moment). The *prolongation formula* [4, 10, 13, 15] is indeed expressing this condition.

We write a vector field in $J^{(k)}M$ as

$$Y \ = \ X \ + \ \sum_{|J|=1}^k \ \Psi^a_J \frac{\partial}{\partial u^a_J}$$

where X is as above, $J = j_1, ..., j_p$ is a multiindex, and the sum is over all multiindices of modulus $|J| = j_1 + ... + j_p$ up to the order of the jet space. We also write D_J for the total derivative $D^{j_1}_{x^1} ... D^{j_p}_{x^p}$, and u^a_J for $D_J u^a$; moreover $u_{J,i}$ will denote $D_i u_J$.

Then Y is the prolongation of X if and only if the coefficients Ψ^a_J satisfy the **prolongation formula**

$$\Psi^a_J \ = \ D_J \varphi^a \ - \ D_J(\xi^i u^a_i) \ + \ \xi^i D_J u^a_i \ . \tag{1}$$

This is also recast in recursive form. We denote by $\widehat{J} = J + e_k$ the multiindex with entries $\widehat{j}_i = j_i + \delta_{ik}$, and for short $u_{J,k} := u_{J+e_k}$, $\Psi^a_{J,k} := \Psi^a_{J+e_k}$. Then (1) is equivalent to

$$\Psi^a_{J,k} \ = \ D_k \Psi^a_J \ - \ u^a_{J,m} D_k \xi^m \tag{2}$$

with $\Psi^a_0 = \varphi^a$ (see e.g. sect. 2.3 of [10]).

Let us now discuss the geometrical aspects of the prolongation operation in terms of contact structures [4, 7, 10, 11, 13]; these will be useful for our subsequent generalization.

Note preliminarly that for any function $f : J^{(k-1)}M \to \mathbf{R}$ we can write

$$\mathrm{d}f \ = \ (D_i f) \mathrm{d}x^i \ + \ \widehat{\vartheta}[f] \tag{3}$$

where $\widehat{\vartheta}[f] \in \mathcal{E}$ is some contact form whose explicit expression (easy to compute) is irrelevant here.

Definition 1. Let Y be a vector field on $J^{(k)}M$. We say that Y **preserves the contact structure** if $\mathcal{L}_Y : \mathcal{E} \to \mathcal{E}$. ♣

Proposition 1. *The vector field $Y \in \mathcal{X}[J^{(k)}M]$, projecting to a vector field $X \in \mathcal{X}[M]$ on M, is the prolongation of a vector field $X \in \mathcal{X}[M]$ if and only if it preserves the contact structure in $J^{(k)}M$.*

Proof. This is a classical result, see e.g. [7, 10, 11, 13]. ♢

Lemma 1. *The vector field Y preserves the contact structure $\mathcal{E}$ if and only if, for any $\vartheta \in \mathcal{E}$ and any $i = 1, ...p$, $([D_i, Y]) \lrcorner \vartheta \ = \ 0$.*

Proof. Write ϑ^a_J and Y as above, and note that $D_i = \partial_i + u^a_{J,i}(\partial/\partial u^a_J)$, where of course $\partial_i = \partial/\partial x^i$. With this notation and standard computations,

$$[D_i, Y] \ = \ (D_i \xi^m) \partial_m \ + \ (D_i \Psi^a_J - \Psi^a_{J,i})(\partial/\partial u^a_J) \ ; \tag{4}$$

hence we get $[D_i, Y] \lrcorner \vartheta^a_J = -\Psi^a_{J,i} + \big(D_i \Psi^a_J - u^a_{J,m}(D_i \xi^m) \big)$, which vanishes if and only if the Ψ^a_J satisfy the recursive prolongation formula (2). ♢

Corollary. *The vector field Y preserves the contact structure $\mathcal{E}$ if and only if $[D_i, Y] = h^m_i D_m + V$ for some $h^m_i \in \Lambda^0(J^{(k)}M)$ and V a vertical vector field for the fibration $\pi_{k,k-1} : J^{(k)}M \to J^{(k-1)}M$.*

Proof. The vector fields D_m span the set of non-vertical vector fields (for the fibration $\pi_{k,k-1}$) in the annihilator of the contact forms. Alternatively, this follows at once from (3), with $h^m_i = D_i \xi^m$. ♢

2 ODEs: λ-prolongations and λ-symmetries

In this section we will restrict to the case of scalar ODEs, i.e. to the case where the bundle (M, π, B) has $B = \mathbf{R}$ as base space and $\pi^{-1}(b) = \mathbf{R}$ as fiber. We will characterize in geometrical terms, i.e. in terms of their action on the contact structure, the λ-prolongations introduced by Muriel and Romero [8] (see also [9] and [5]), and further studied by Pucci and Saccomandi [12].

We simply write u_n for $D^n_x u$, and similarly for Ψ_n. The standard contact forms in $J^{(k)}M$ will be $\vartheta_n = \mathrm{d}u_n - u_{n+1}\mathrm{d}x$, with $n = 0, ..., k-1$.

We start by recalling the definition of λ-prolongations and λ-symmetries as given by Muriel and Romero, using an obvious notation for x-derivatives of the u.

Definition 2. Let $X = \xi(\partial/\partial x) + \varphi(\partial/\partial u)$ be a vector field on M, and $Y = X + \sum_{n=1}^{k} \Psi_n(\partial/\partial u_n)$ a vector field on $J^{(k)}M$. Let $\lambda : J^{(1)}M \to \mathbf{R}$ be a smooth function. We say that Y is the λ-*prolongation* of X if its coefficients satisfy the **λ-prolongation formula**

$$\Psi_{n+1} \ = \ [(D_x + \lambda)\Psi_n] \ - \ u_{n+1}[(D_x + \lambda)\xi] \tag{5}$$

for all $n = 0, ..., k-1$. ♣

Definition 3. Let Δ be a k-th order ODE for $u = u(x)$, $u \in U = \mathbf{R}$, and let $(M = U \times B, \pi, B)$ be the corresponding variables bundle. Let the vector field Y in $J^{(k)}M$ be the λ-prolongation of the Lie-point vector field X in M. Then we say that X is a **λ-symmetry** of Δ if and only if Y is tangent to the solution manifold S_Δ, i.e. iff there is a smooth function Φ on $J^{(k)}M$ such that $Y(\Delta) = \Phi\Delta$. ♣

Remark 1. We stress that in this note we take $\lambda : J^{(1)}M \to \mathbf{R}$, which guarantees that the λ prolongation of a Lie-point vector field in M is a proper vector field in each $J^{(n)}M$. One could also consider $\lambda : J^{(r)}M \to \mathbf{R}$, obtaining obvious generalizations of the results given here. In this case the λ-prolongations of X would be generalized vector fields in each $J^{(n)}M$ with $n > 0$ even if X is a Lie-point vector field. The same applies to the μ-prolongations to be considered in later sections. ⊙

We will not discuss here the relevance of λ-symmetries, referring to [8, 12]; we just recall that they are as useful as standard ones in that one can perform symmetry reduction to the same extent as for standard symmetries [8].

The basic property of λ-prolongations behind this feature was clearly pointed out by Pucci and Saccomandi [12], and can be expressed in terms of the characteristics of the vector fields Y which are λ-prolongations of X.

Given the vector bundle (M, π, B), we choose a distinguished smooth real function $\lambda(x, u, u_x) : J^{(1)}M \to \mathbf{R}$. We note for later discussion that to this is associated a semibasic one-form $\mu \in \Lambda^1(J^{(1)}M)$, i.e. the one-form $\mu = \lambda(x, u, u_x)\mathrm{d}x$

Definition 4. Let Y be a vector field on the contact manifold $(J^{(k)}M, \mathcal{E})$, and $\lambda \in \Lambda^0(J^{(1)}M)$ a smooth function on M. We say that Y **λ-preserves the contact structure** if, for any contact one-form $\theta \in \mathcal{E}$,

$$\mathcal{L}_Y(\theta) \ + \ (Y \lrcorner \theta)\,\lambda \mathrm{d}x \ = \ \widehat{\theta} \tag{6}$$

for some contact one-form $\widehat{\theta} \in \mathcal{E}$. ♣

Theorem 1. *Let (M, π, B) be a bundle over the real line $B = \mathbf{R}$ with fiber $\pi^{-1}(x) = \mathbf{R}$, and let $\mathcal{E}$ be the standard contact structure in $J^{(k)}M$. Let Y be a vector field on the jet space $J^{(k)}M$, which projects to a vector field X on*

M. Then Y is the λ-prolongation of X if and only if it λ-preserves the contact structure.

Proof. We write a general vector field on $J^{(k)}M$ as $Y = \xi\partial_x + \sum_{m=0}^k \Psi_m(\partial/\partial u_m)$; as the contact forms are $\vartheta_n = \mathrm{d}u_n - u_{n+1}\mathrm{d}x$ $(n = 0, ..., k-1)$, we have by explicit computation

$$\mathcal{L}_Y(\vartheta_n) + (Y \lrcorner \vartheta_n)\lambda\mathrm{d}x = [-\Psi_{n+1} + D_x\Psi_n - u_{n+1}D_x\xi + \lambda(\Psi_n - u_{n+1}\xi)]\,\mathrm{d}x + \widehat{\theta}$$

with $\widehat{\theta}$ a contact form. Thus (6) is satisfied if and only if the Ψ_n satisfy the λ-prolongation formula (5). $\diamondsuit$

We can also provide an alternative characterization of λ-prolonged vector fields, similarly to what we did for standard prolongations in lemma 1.

Lemma 2. *Let Y be a vector field on the jet space $J^{(k)}M$, with (M, π, B) a vector bundle over the real line $B = \mathbf{R}$, and let $\mathcal{E}$ be the standard contact structure in $J^{(k)}M$. Then Y is the λ-prolongation of a vector field X on M if and only if, for any $\vartheta \in \mathcal{E}$,*

$$[D_x, Y] \lrcorner \vartheta \ = \ \lambda(Y \lrcorner \vartheta) \ . \tag{7}$$

Proof. Looking at the proof of lemma 1, $[D_x, Y] \lrcorner \vartheta$ is given by (4) specialized to the case $p = 1$: with the obvious notation $u_n := D_x^n u$ (and similarly for Ψ_n) we have $[D_x, Y] = -\Psi_{n+1} + (D_x\Psi_n - u_{n+1}D_x\xi)$; on the other hand, it is easy to check that $Y \lrcorner \vartheta_n \ = \ \Psi_n - u_{n+1}\xi$. Thus eq. (7) is equivalent to $\Psi_{n+1} = [(D_x + \lambda)\Psi_n] - u_{n+1}[(D_x + \lambda)\xi]$, i.e. to the λ-prolongation formula (5). $\diamondsuit$

Corollary. *In the hypotheses of lemma 2, Y is the λ-prolongation of a vector field X on M, if and only if $[D_x, Y] = \lambda Y + hD_x + V$ with λ, h scalar functions on $J^{(1)}M$ and V a vertical vector field for the fibration $\pi_{k,k-1} : J^{(k)}M \to J^{(k-1)}M$.*

Remark 2. Theorem 1 shows that our geometrical formulation, i.e. definition 4, is equivalent to the standard (analytical) one, i.e. definition 2. The advantage of our formulation is twofold: we have a geometrical characterization of λ-prolongations (λ-symmetries), and moreover this is readily extended from the ODEs to the PDEs case. As we discuss later on, we can moreover generalize the standard symmetry reduction method for PDEs to an analogous λ-symmetry reduction. We will also show that this definition and the reduction procedure extend to systems of PDEs. $\odot$

3 PDEs: μ-prolongations and μ-symmetries

In this section we extend our approach to λ-prolongations and λ-symmetries to the case of scalar PDEs (p independent variables); the case of PDE systems will be dealt with in section 5 below.

The role of the scalar function λ will now be played by an array of p smooth functions $\lambda_i : J^{(1)}M \to \mathbf{R}$ (remark 1 holds also in this context), which will be the components of a semibasic form $\mu \in \Lambda^1(J^{(1)}M)$. The only additional ingredient required in the multi-dimensional (PDE) case is a compatibility condition between the semibasic form μ and the contact structure – this is eq.(10) below – automatically satisfied in the ODE case.

Actually, our formulation of λ-prolongations in the ODE case was such that the results, and even their proofs, are the same also in the PDE case – except of course for the appearance of new variables.

In view of our geometric approach it is convenient to focus on the form μ rather than on the q-ple of smooth functions λ_i. We will thus call the analogue of λ-prolongations and λ-symmetries in the PDE frame, μ-prolongations and μ-symmetries.

We equip $(J^{(1)}M, \pi, B)$ with a distinguished semibasic one-form μ,

$$\mu \ = \ \lambda_i \, \mathrm{d}x^i \ . \tag{8}$$

We require that μ is *compatible* with the contact structure defined in $J^{(k)}M$, for $k \geq 2$, in the sense that

$$\mathrm{d}\mu \in J(\mathcal{E}) \ , \tag{9}$$

where $J(\mathcal{E})$ is the Cartan ideal generated by $\mathcal{E}$ (we recall that a two-form α is in $J(\mathcal{E})$ if and only if $\alpha = \rho^J \wedge \vartheta_J$ for some one-forms ρ^J).

It should be noted that this condition does not appear when we deal with first order equations, i.e. with first order μ-prolongations. We note also that for $p = 1$ eq.(9) is automatically satisfied: indeed, $\mathrm{d}\mu = (\partial\lambda/\partial u)\mathrm{d}u \wedge \mathrm{d}x + (\partial\lambda/\partial u_x)\mathrm{d}u_x \wedge \mathrm{d}x = (\partial\lambda/\partial u)\vartheta_0 \wedge \mathrm{d}x + (\partial\lambda/\partial u_x)\vartheta_1 \wedge \mathrm{d}x$.

Lemma 3. *Condition (9) is equivalent to*

$$D_i\lambda_j - D_j\lambda_i \ = \ 0 \ . \tag{10}$$

This is in turn equivalent to the condition that the operators $\nabla_i := D_i + \lambda_i$ commute, $[\nabla_i, \nabla_j] = 0$.

Proof. As λ_i is a function on $J^{(1)}M$, we have $\mathrm{d}\mu = (\partial\lambda_j/\partial x^i)\mathrm{d}x^i \wedge \mathrm{d}x^j + (\partial\lambda_j/\partial u)\mathrm{d}u \wedge \mathrm{d}x^j + (\partial\lambda_j/\partial u_i)\mathrm{d}u_i \wedge \mathrm{d}x^j$, i.e.

$$\begin{aligned} \mathrm{d}\mu = \ & [(\partial\lambda_j/\partial x^i) + u_i(\partial\lambda_j/\partial u) + u_{ik}(\partial\lambda_j/\partial u_k)]\mathrm{d}x^i \wedge \mathrm{d}x^j + \\ & + (\partial\lambda_j/\partial u)\vartheta_0 \wedge \mathrm{d}x^j \ + \ (\partial\lambda_j/\partial u_i)\vartheta_i \wedge \mathrm{d}x^j \ . \end{aligned}$$

The two latter terms are of course in $J(\mathcal{E})$, while no form $\mathrm{d}x^i \wedge \mathrm{d}x^j$ belongs to $J(\mathcal{E})$; thus, (9) is satisfied if and only if the coefficients of all these terms vanish. This condition is precisely (10). (Note this extends to the case where μ is semibasic for $(J^{(n)}, \pi_n, B)$, see remark 1.) The equivalence of this with $[\nabla_i, \nabla_j] = 0$ follows from the definition of ∇_i. $\diamondsuit$

Definition 5. Let Y be a vector field on the contact manifold $(J^{(k)}M, \mathcal{E})$, and μ a semibasic form on M compatible with $\mathcal{E}$. We say that Y **μ-preserves the**

contact structure if, for any $\theta \in \mathcal{E}$, there is a form $\widehat{\theta} \in \mathcal{E}$ such that

$$\mathcal{L}_Y(\theta) \ + \ (Y \lrcorner \theta)\mu \ = \ \widehat{\theta} \ . \tag{11}$$

♣

Definition 6. A vector field Y in $J^{(k)}M$ which projects to X in M and which μ-preserves the contact structure is said to be the **μ-prolongation** of order k, or the k-th μ-prolongation, of X. ♣

Theorem 2. *Let Y be a vector field on the jet space $J^{(k)}M$, with (M,π,B) a vector bundle over $B = \mathbf{R}^p$, written in coordinates as*

$$Y \ = \ X + \sum_{|J|=1}^{k} \Psi_J \frac{\partial}{\partial u_J} \ ,$$

with $X = \xi^i(\partial/\partial x^i) + \varphi(\partial/\partial u)$ a vector field on M. Let $\mathcal{E}$ be the standard contact structure in $J^{(k)}M$, and $\mu = \lambda_i \mathrm{d}x^i$ a semibasic one-form on $(J^{(1)}M,\pi,B)$, compatible with $\mathcal{E}$. Then Y is the μ-prolongation of X if and only if its coefficients (with $\Psi_0 = \varphi$) satisfy the **μ-prolongation formula**

$$\Psi_{J,i} \ = \ (D_i + \lambda_i)\Psi_J \ - \ u_{J,m}\,(D_i + \lambda_i)\xi^m \ . \tag{12}$$

Proof. The standard contact forms in $J^{(k)}M$ are $\vartheta_J = \mathrm{d}u_J - u_{J,i}\mathrm{d}x^i$, with $|J| = 0,...,k-1$. Thus, as already computed in the proof of proposition (1), $\mathcal{L}_Y(\vartheta_J) = (-\Psi_{J,i} + D_i\Psi_J - u_{J,m}D_i\xi^m)\mathrm{d}x^i + \Theta$ with Θ a contact form. On the other hand, it is easy to compute that $Y \lrcorner \vartheta_J = \Psi_J - u_{J,m}\xi^m$. Therefore,

$$\begin{aligned} & \mathcal{L}_Y(\vartheta_J) + (Y \lrcorner \vartheta_J)\mu \ = \\ & \quad = \ [(-\Psi_{J,i} + D_i\Psi_J - u_{J,m}D_i\xi^m) + \lambda_i(\Psi_J - u_{J,m}\xi^m)]\,\mathrm{d}x^i \ + \ \Theta \ . \end{aligned}$$

This is a contact form if and only if the coefficients of all the $\mathrm{d}x^i$ vanish, i.e. if and only if (12) is satisfied. ♢

Remark 3. Condition (9) arises from the following: consider the multiindices $J = (j_1,...,j_p)$ and $L = (\ell_1,...,\ell_p)$ with $\ell_s = j_s + \delta_{i,s} + \delta_{k,s}$; the coefficient Ψ_L can be obtained from Ψ_J by applying twice formula (12), but we can proceed in two different ways, i.e. pass first from Ψ_J to $\Psi_{J,i}$ and then to Ψ_L, or pass first from Ψ_J to $\Psi_{J,k}$ and then to Ψ_L. Needless to say, the result must be the same in the two cases, and this is the **compatibility condition** for the λ_i. By explicit computation this is just (10), equivalent to (9) by lemma 3. ⊙

As for standard and λ-prolongations, μ-prolongations have a specific behaviour for what concerns their commutation with the total derivatives D_i.

Lemma 4. *If Y is the μ-prolongation of a Lie-point vector field X, with $\mu = \lambda_i \mathrm{d}x^i$, then for any contact form ϑ,*

$$[D_i, Y] \lrcorner \vartheta \ = \ \lambda_i\,(Y \lrcorner \vartheta) \ . \tag{13}$$

Proof. In the proof of lemma 1 we have computed $[D_i, Y] \lrcorner \vartheta_J = -\Psi_{J,i} + D_i \Psi_J - u_{J,m} D_i \xi^m$; needless to say, $Y \lrcorner \vartheta_J = \Psi_J - u_{J,m}\xi^m$ and thus (13) is equivalent to the μ-prolongation formula (12). $\diamondsuit$

Corollary. *In the hypotheses of lemma 4, Y is the μ-prolongation of a vector field X on M, if and only if $[D_i, Y] = \lambda_i Y + h_i^m D_m + V$ with λ_i, h_i^m scalar functions on $J^{(1)}M$ and V a vertical vector field for the fibration of $J^{(k)}M$ over $J^{(k-1)}M$.*

It is quite remarkable that a simple relation exists between the μ-prolongation of a vector field and its ordinary prolongation. In order to discuss this relation, we write $X = \xi^i(\partial/\partial x^i) + \varphi(\partial/\partial u)$ for the vector field in M, and denote its ordinary prolongations as $X^{(k)} = X + \widetilde{\Psi}_J(\partial/\partial u_J)$, while its μ-prolongations are denoted as $Y = X + \Psi_J(\partial/\partial u_J)$. The form μ is written, as usual, $\mu = \lambda_i \mathrm{d}x^i$, and of course $\Psi_J = \widetilde{\Psi}_J$ when all the λ_i (or at least all those for i such that $j_i \neq 0$) vanish.

As well known [4, 10, 13, 15], the vector field X can be cast in evolutionary form as $X_Q := Q(\partial/\partial u)$, with $Q := \varphi - u_i \xi^i$.

The equations $D_J Q = 0$, with $|J| = 0, ..., k-1$ identify the X-invariant space $\mathcal{I}_X \subset J^{(k)}M$. We denote by $\mathcal{F}$ the module over $\mathcal{C}^\infty(J^{(k)}M)$ generated by the $D_J Q$, i.e. the set of functions F which can be written as $F = c^J D_J Q$ for some smooth functions $c^J : J^{(k)}M \to \mathbf{R}$, and by $\mathcal{F}^{(m)} \subseteq \mathcal{F} \equiv \mathcal{F}^{(k)}$ those which depend only on variables $(x, u^{(m)})$, $m \leq k$. Needless to say, $D_i : \mathcal{F}^{(m-1)} \to \mathcal{F}^{(m)}$.

Theorem 3. *Let X, Y, μ be as above. Write Ψ_J as $\Psi_J = \widetilde{\Psi}_J + F_J$. Then the functions F_J satisfy the recursion relation (with $F_0 = 0$)*

$$F_{J,i} \ = \ (D_i + \lambda_i) F_J \ + \ \lambda_i D_J Q \ . \qquad (14)$$

Proof. In order to show that the statement of the theorem holds at all orders, we proceed recursively: we suppose (14) holds for all $|J| < h$, and wish to prove that it holds also for $|J| = h$. Any $\widehat{J}$ of order h can be written as $J + e_i$ for some i and some J of order $h-1$; formula (14) holds for Ψ_J. Thus, by the μ-prolongation formula,

$$\begin{aligned} \Psi_{J,i} \ &= \ (D_i + \lambda_i)\widetilde{\Psi}_J - u_{J,m}(D_i + \lambda_i)\xi^m + (D_i + \lambda_i)F_J \\ &= \ [D_i \widetilde{\Psi}_J - u_{J,m} D_i \xi^m] + \lambda_i [\widetilde{\Psi}_J - u_{J,m}\xi^m] + (D_i + \lambda_i) F_J \ . \end{aligned}$$

The first term is just $\widetilde{\Psi}_{J,i}$, and the last is already in the form appearing in (14); so we have to look only at the second one.

Take an s such that $j_s \neq 0$ in the multiindex J, and write $K = J - e_s$. Then, using the standard prolongation formula,

$$\widetilde{\Psi}_J - u_{J,m}\xi^m = [D_s \widetilde{\Psi}_K - u_{K,m} D_s \xi^m] - (D_s u_{K,m})\xi^m = D_s \left(\widetilde{\Psi}_K - u_{K,m}\xi^m \right) \ .$$

We can then repeat the procedure on any index q such that $K = J - e_s$ has a nonzero q entry, and so on. In the end, recalling that $\Psi_0 = \varphi$, we have

$$\widetilde{\Psi}_J - u_{J,m} \xi^m \ = \ D_J \left(\varphi - u_m \xi^m \right) \ = \ D_J Q \ ;$$

this also follows from the formula for prolongation of the evolutionary representative of X. Going back to our computation, we have thus shown that

$$\Psi_{J,i} \ = \ \widetilde{\Psi}_{J,i} \ + \ \lambda_i D_J Q \ + \ (D_i + \lambda_i) F_J \ .$$

This shows that if (14) is satisfied at order $h-1$, it is also satisfied at order h.

It is easy to check that (14) holds at order one, i.e. for $|J| = 0$: indeed, by the μ-prolongation formula (12) and the ordinary prolongation formula (1),

$$\begin{aligned} \Psi_i \ &= \ (D_i + \lambda_i)\varphi - u_m (D_i + \lambda_i) \xi^m \\ &= \ (D_i \varphi - u_m D_i \xi^m) \ + \ \lambda_i \left(\varphi - u_m \xi^m \right) \\ &= \ \widetilde{\Psi}_i + \lambda_i Q \ . \end{aligned}$$

We conclude that (14) holds at all orders. $\diamondsuit$

This theorem provides an economic way of computing μ-prolongations of X if we already know its ordinary prolongations. Theorem 3 also has a rather obvious consequence, which will be relevant in the following.

Lemma 5. *Let X be a vector field on M, $\mathcal{E}$ the standard contact structure on $J^{(k)}M$, and μ any semibasic form on M compatible with $\mathcal{E}$. Then:* (i) *the μ-prolongation Y of X coincides with the ordinary prolongation $X^{(k)}$ on the invariant space $\mathcal{I}_X$;* (ii) *the space $\mathcal{I}_X \subset J^{(k)}M$ is invariant under the μ-prolongations of X, for any semibasic form μ compatible with $\mathcal{E}$.*

Proof. By definitions, any function $F \in \mathcal{F}$ vanishes identically on $\mathcal{I}_X$. Thus (14) guarantees that $\Psi_J = \widetilde{\Psi}_J$ on $\mathcal{I}_X$, i.e. proves point *(i)*.

As for *(ii)*, this is a known property of standard prolongations, easily checked by using the evolutionary representative of X, $X_Q := Q (\partial / \partial u)$. Its prolongation is $X_Q^{(k)} = (D_J Q)(\partial / \partial u_J)$, where the sum is over all multiindices with $|J| \leq k$, and $X^{(k)} = X_Q^{(k)} + \xi^i D_i$. Thus, $X^{(k)}$ reduces to $W = \xi^i D_i$ on $\mathcal{I}_X$; and W is obviously tangent to $\mathcal{I}_X$. $\diamondsuit$

Finally, we define μ-symmetries of a PDE as Lie-point vector fields whose μ-prolongation is a symmetry of the equation.

Definition 7. Let X be a vector field on M, and let $Y \in \mathcal{X}[J^{(k)}M]$ be its μ-prolongation of order k. Let Δ be a differential equation of order k in M, $\Delta := F(x, u^{(k)}) = 0$, and $\mathcal{S} \subset J^{(k)}M$ be the solution manifold for Δ. If $Y : \mathcal{S} \to \mathrm{T}\mathcal{S}$, we say that X is a **μ-symmetry** for Δ. If Y leaves invariant each level manifold for F, we say that X is a **strong μ-symmetry** for Δ. $\clubsuit$

Remark 4. Note that if we look for μ-symmetries of a given equation Δ, we can accept forms μ which do not satisfy (9) on the whole jet space $J^{(n)}M$, but only on the solution submanifold $S_\Delta \subset J^{(n)}M$. $\odot$

Remark 5. Given a form $\mu = \lambda_i \mathrm{d}x^i$, we consider exponential vector fields

$$X \ = \ e^{\int \mu} \cdot \ X_0$$

where X_0 is a vector field on M; note that if $\mu = (D_i P)\mathrm{d}x^i$, in which case (10) is automatically satisfied, then $X = e^P X_0$; in general X is a formal expression. For a general μ, consider an equation Δ such that (10) is satisfied on S_Δ, see the remark above. Then we have the following result (see [2] for a proof and extensions): X is a (in general, nonlocal) symmetry for Δ if and only if X_0 is a μ-symmetry for Δ. This extends a result by Muriel and Romero [8]. $\odot$

The relevant point is that **μ-symmetries can be used to obtain group-invariant solutions**, i.e. one can introduce μ-symmetry reductions of PDEs and obtain invariant solutions to the original PDE from these, by the same method as for standard symmetries.

Note that in this way we parallel again the ODE case, where it was proven by Muriel and Romero and by Pucci and Saccomandi that λ-symmetries are as good as standard ones for reduction of the equation.

4 The μ-symmetry reduction method for PDEs

As well known, symmetry reduction for PDEs is conceptually different from symmetry reduction for ODEs: while in the latter case it yields a reduced equation whose solutions provide, together with an integration, the most general solution to the original ODE, in the PDE case the reduced equation provides only the symmetry-invariant solutions to the original PDE.

4.1 The PDE reduction method

In this subsection we briefly recall (using the notation introduced so far) symmetry reduction for scalar PDEs in the case of standard symmetries; this is discussed in detail in a number of textbooks and research papers, see e.g. [4, 10, 13, 15]. We will just discuss reduction under a single vector field, rather than a general (i.e. higher dimensional) Lie algebra.

Consider a PDE of order k Δ, which we may think in the form $F(x, u^{(k)}) = 0$ with $F : J^{(k)}M \to \mathbf{R}$ a smooth scalar function. Let the Lie-point vector field X in M, with prolongation $X^{(k)}$ in $J^{(k)}M$, be a (standard) symmetry for Δ. Then we proceed as follows, following Olver. (For more details, see e.g. the discussion in chapter 3 of [10]).

First of all we pass to symmetry-adapted coordinates in M. In practice, we have to determine a set of p independent invariants for X in M, which we will denote as $(y^1, ..., y^{p-1}, v)$: these will be our X-invariant coordinates, and essentially identify the G-orbits, while the remaining coordinate σ will be acted upon by G. In other words, G-orbits will correspond to fixed value of (y, v) coordinates and to σ taking values in a certain subset of the real line (thus (y, v) are coordinates on the *orbit space* $\Omega = M/G$, see [10])

The invariants will be given by some functions $y^i = \eta^i(x,u)$ $(i=1,...,p-1)$ and $v = \zeta(x,u)$ of $x^1,...,x^p$ and u. If X acts transversally, we can invert these for x and u as functions of $(y,v;\sigma)$, i.e. write $x^i = \chi^i(y,v;\sigma)$ $(i=1,...,p)$ and $u = \beta(y,v;\sigma)$.

If now we decide to see the $(y;\sigma)$ as independent variables and the v as the dependent one, we can use the chain rule to express x-derivatives of u as σ and y-derivatives of v[1]. Using these, we can finally write Δ in terms of the $(y,v;\sigma)$ coordinates and derivatives of v in the y and σ; this will turn out to be an equation which, when subject to the side condition $\partial v/\partial\sigma = 0$, is independent of σ. The condition $\partial v/\partial\sigma = 0$ expresses the fact that the solutions are required to be invariant under X, i.e. the equation obtained in this way represents the restriction of Δ to the space of G-invariant functions, and therefore it is sometimes also denoted as Δ/G.

Suppose we are able to determine some solution $v = \Phi(y)$ to the reduced equation; we can write this in terms of the (x,u) coordinates as $\zeta(x,u) = \Phi[\eta(x,u)]$, which yields implicitly $u = f(x)$: this is the corresponding X-invariant solution to the original equation Δ in the original coordinates.

Remark 6. The symmetry reduction method for PDEs can also be seen in a slightly different way: if we look for X-invariant solution $u = f(x)$ to Δ, we determine the characteristic $Q = \varphi - u_i\xi^i$ of the vector field X, and supplement Δ with the equations $E_J := D_J Q = 0$ with $|J| = 0,...,k-1$. The equation E_0 requires that the evolutionary representative $X_Q = Q(\partial/\partial u)$ vanish on γ_f, i.e. that u is X-invariant, and all the equations with $|J| > 0$ are just differential consequences of this. The X-invariant solutions to Δ are in one to one correspondence with the solutions to the system $\Delta_{(X)} := \{\Delta; E_J\}$. See e.g. [15] for details, and for how this approach is used in a more general context. ⊙

Remark 7. We stress that the standard method discussed here applies under a nondegeneracy (transversality) condition, guaranteeing a certain Jacobian admits an inverse. When this is not the case – as it happens in a number of physically relevant cases – the treatment should go through the approach developed by Anderson, Fels and Torre [1]. See also [6] for the case of partial transversality. We also stress that this method is justified only if the (possibly, only local) one-parameter group G generated by X has regular action in M, i.e. the G-orbits are regular embedded submanifolds of M [3]. In the following we tacitly assume both conditions mentioned here are satisfied. ⊙

4.2 On the justification of the method

The method described above is rigorously justified in chapter 3 of [10], to which we refer for details. In this subsection we just recall what is the key step in the

[1] It is maybe worth recalling that this computation can be described also in a slightly different, but equivalent, way: that is, we write $\mathrm{d}u = u_i \mathrm{d}x^i$ on the one hand, and $\mathrm{d}u = \mathrm{d}[\beta(y,v;\sigma)]$ on the other. We then expand the latter as $\mathrm{d}u = \beta_j \mathrm{d}y^j + \beta_v \mathrm{d}v + \beta_\sigma \mathrm{d}\sigma$, where of course $\beta_j = \partial\beta/\partial y^j$, and substitute for $\mathrm{d}v$ as $\mathrm{d}v = v_j \mathrm{d}y^j + v_\sigma \mathrm{d}\sigma$. Comparing the two expressions for $\mathrm{d}u$, we obtain the expression for u_i in terms of v_j and v_σ.

proof, as we will have to prove a similar property also holds for μ-prolongations in order to justify the extension of this method to μ-symmetries.

We recall that a function $u = f(x^1, ..., x^p)$ corresponds to a section $\gamma_f \in \Gamma[M]$, the space of sections for the bundle (M, π, B), i.e. $\gamma_f = \{(x, u) : u = f(x)\}$. This is uniquely prolonged to a section $\gamma_f^{(k)} \in \Gamma[J^{(k)}M]$; $\gamma_f^{(k)}$ is the unique lift of the curve γ_f in M to a curve in $J^{(k)}M$ which (i) projects down to γ_f in M, and (ii) is everywhere tangent to the field of contact linear spaces.

When we act with a vector field $X = \xi^i(\partial/\partial x^i) + \varphi(\partial/\partial u)$ on M, at the infinitesimal level, the section γ_f is mapped into $\gamma_{\widehat{f}}$ with

$$\widehat{f}(x) \ = \ f(x) \ + \ \varepsilon[\varphi(x,u) - \xi^i(x,u)\partial_i f(x)]_{u=f(x)} \ + \ o(\varepsilon) \ .$$

Thus a function $u = f(x)$ is invariant under the action of the vector field X in M if and only if $\widehat{Q}(x) := Q[x, f(x)] = 0$ (with Q the characteristic of X).

We consider the equation $E_0 := Q = 0$ and all of its differential consequences $E_J := D_J Q = 0$ for $|J| < k$; this identifies the invariant manifold $\mathcal{I}_X \subset J^{(k)}M$. Passing to the evolutionary representative $X_Q = Q(\partial/\partial u)$ of X, it is obvious that X_Q and its prolongations vanish on $\mathcal{I}_X$. The X-invariant solutions to Δ will be the solutions to the system $\Delta_{(X)}$ made of Δ and of the invariance condition:

$$\begin{cases} F(x, u^{(k)}) = 0 \\ D_J Q = 0 \qquad\qquad (|J| = 0, ..., k-1) \ . \end{cases} \tag{15}$$

We denote the solution manifold to this system as $\mathcal{S}_X \subset \mathcal{I}_X \subset J^{(k)}M$. The invariance of $\mathcal{S}_X$, as discussed by Olver [10], guarantees that the method recalled above is justified.

Recall now that the prolongations of X and X_Q satisfy

$$X^{(k)} \ = \ X_Q^{(k)} \ + \ \xi^i D_i^{(k)} \ . \tag{16}$$

Lemma 6. *The (standard) prolongation $X^{(k)}$ of X reduces to $\xi^i D_i$ on $\mathcal{I}_X$, and is tangent to $\mathcal{S}_X$.*

Proof. The field $X_Q^{(k)}$ vanishes on $\mathcal{I}_X$ because of the equations E_J, and the D_i are symmetries of any system, as the differential consequences of any equation of the system are satisfied by solutions to the system. By (16), this proves the claim. $\diamondsuit$

4.3 Reduction of PDEs under μ-symmetries

In the case of μ-symmetries of PDEs, we can proceed exactly in the same way as for standard symmetries in order to determine G-invariant solutions.

Note that the step consisting in the introduction of symmetry-adapted coordinates is exactly the same; the difference lies of course in the step connected to the prolongation structure.

We describe here how the standard symmetry reduction method is formulated to deal with μ-symmetries. We suppose that X is a μ-symmetry of Δ, acting transversally for the fibration (M, π, B), and denote the μ-prolongation of X as $Y \in \mathcal{X}[J^{(k)}M]$.

First of all we pass to symmetry-adapted coordinates $(y, v; \sigma)$ in M, as in the standard case. We retain the notation introduced in subsection 1. We further proceed as there, i.e. use the chain rule to express x-derivatives of the u as σ and y-derivatives of the v. Using these, we can finally write Δ in terms of the $(y, v; \sigma)$ coordinates and their derivatives.

Again, looking for X-invariant solutions means supplementing the equation with the side condition $\partial v/\partial\sigma = 0$, or with the conditions $D_J Q = 0$, see eq.(15) above, in the original coordinates.

Now the point is that *if* the equation thus obtained is independent of σ, we have indeed obtained a symmetry reduction of the original equation. In this case solutions $v = \Phi(y)$ to the reduced equation can be written in terms of the (x, u) coordinates as $\zeta(x, u) = \Phi[\eta(x, u)]$ and yield implicitly $u = f(x)$, the corresponding X-invariant solution to the original equation.

However, the vector field Y is not the ordinary prolongation of X, and thus we are not *apriori* guaranteed it leaves $\mathcal{S}_X$ or $\mathcal{I}_X$ invariant. Thus, in order to justify the method sketched above – i.e. in order to prove that the standard PDE reduction method still applies in the case of μ-symmetries – we have to prove the following theorem 4. Note that the only difference with respect to the standard case will be that it is the vector field Y, and not the ordinary prolongation $X^{(k)}$ of X, to be tangent to the solution manifold of Δ in $J^{(k)}M$.

Theorem 4. *Let Δ be a scalar PDE of order k for $u = u(x^1, ..., x^p)$. Let $X = \xi^i(\partial/\partial x^i) + \varphi(\partial/\partial u)$ be a vector field on M, with characteristic $Q := \varphi - u_i\xi^i$, and let Y be the μ-prolongation of order k of X. If X is a μ-symmetry for Δ, then $Y : \mathcal{S}_X \to \mathrm{T}\mathcal{S}_X$, where $\mathcal{S}_X \subset J^{(k)}M$ is the solution manifold for the system Δ_X made of Δ and of $E_J := D_J Q = 0$ for all J with $|J| = 0, ..., k-1$.*

Proof. Recall that $\mathcal{S}_X$ is the intersection of the solution manifold $\mathcal{S}_0$ to Δ with the X-invariant set $\mathcal{I}_X$ (see remark 3 above, or [15]). The former is Y-invariant by assumption, as X is a μ-symmetry of Δ; the Y-invariance of $\mathcal{I}_X$ is guaranteed by lemma 5 above. Therefore the proof for the standard case [10] extends to the present setting. $\diamondsuit$

Remark 8. The property $Y : \mathcal{I}_X \to \mathrm{T}\mathcal{I}_X$ can be shown in a alternative way without resorting to comparison with the standard case, i.e. using the geometrical characterization of μ-prolonged vector fields, as follows.

Denote by $\mathcal{I}_X^{(m)} \subset J^{(k)}M$ the set of points identified by E_J for $|J| \le m$. We first show that if $\mathcal{I}_X^{(m)}$ is invariant under Y, then $\mathcal{I}_X^{(m+1)}$ is also Y-invariant (for $m = 0, ..., k-2$). Note that Y-invariance of $\mathcal{I}_X^{(m)}$ means that for all $|J| \le m$ there are functions β^K such that $Y(D_J Q) = \sum_{|K|=0}^{m} \beta^K D_K Q$.

We have $Y[D_i(D_J Q)] = [Y, D_i](D_J Q) - D_i(Y(D_J Q))$; from the corollary to lemma 4 this reads $\lambda_i Y(D_J Q) + h_i^s D_s(D_J Q) - D_i(Y(D_J Q)) + V(D_J Q)$, with $V = \sum_{|K|=k} \ell^K(\partial/\partial u_K)$. The first term is in $\mathcal{I}_X^{(m)}$ by hypothesis, while the

second and third ones are by definition in $\mathcal{I}_X^{(m+1)}$. The last term vanishes since $D_J Q$ does not contain u derivatives of order greater than $m+1$, and $m \leq k-2$.

The proof of Y-invariance of $\mathcal{I}_X$ is hence reduced to proving Y-invariance of $\mathcal{I}_X^{(0)}$, i.e. of the manifold identified by $Q = 0$; as for X a Lie-point vector field Q depends only on first order derivatives, it suffices to consider the first μ-prolongation of X, which is just $X^{(1)} + \lambda_i Q \partial_{u_i}$. It is well known that $Q = 0$ is invariant under the ordinary prolongation $X^{(1)}$, and of course the other term vanishes on $Q = 0$.

This proves Y-invariance of $\mathcal{I}_X^{(0)}$ and hence, by the recursive argument given above, of all the $\mathcal{I}_X^{(m)}$ with $m = 0, 1, ..., k-1$.

The recursive property considered here can be seen as a counterpart in the PDE case to the recursive property discussed by Pucci and Saccomandi as characterizing the λ-prolongations as telescopic vector fields in the ODE case [2]. $\odot$

5 Systems of PDEs

In this section we extend μ-prolongations to the case of $q > 1$ dependent variables. We will assume that the dependent variables u take value in the vector space $U = \mathbf{R}^q$, and $B = \mathbf{R}^p$.

It will be natural in this context to consider differential forms taking values in the space $\mathcal{G} = g\ell(q)$, the Lie algebra of the group $G = GL(q)$. Thus we will deal with *matrix-valued differential forms* (or more generally Lie-algebra valued differential forms), see e.g. [14]. The form μ will now be written in local coordinates as

$$\mu \ := \ (\Lambda_i)^a_b \, \mathrm{d}x^i \tag{17}$$

where $\Lambda_i : J^{(1)}M \to \mathcal{G}$ are smooth q-dimensional real matrix functions. Note that remark 1 applies also to this case.

5.1 μ-prolongations in vector framework

In this case we generalize condition (6) to the following (19); we will then define μ-prolongation in the same way as in the scalar case.

In the vector case, we see the contact structure Θ (we use a different symbol than in the scalar case to emphasize we deal with vector-valued forms) as spanned by vector-valued one-forms $\vartheta_J = (\vartheta^1_J, ..., \vartheta^q_J) \in \mathbf{R}^q \otimes \Lambda^1(M)$, where

$$\vartheta^a_J \ = \ \mathrm{d}u^a_J \ - \ u^a_{J,m} \mathrm{d}x^m \ . \tag{18}$$

Definition 5'. We say that Y **μ-preserves the contact structure** Θ, with μ given by (17), if for any vector-valued contact forms $\vartheta \in \Theta$, there is a vector-valued contact forms $\widehat{\vartheta} \in \Theta$ such that

$$\mathcal{L}_Y(\vartheta^a) \ + \left(Y \lrcorner \left[(\Lambda_i)^a_b \vartheta^b\right]\right) \ \mathrm{d}x^i \ = \ \widehat{\vartheta}^a \ . \tag{19}$$

♣

Definition 6'. A vector field Y in $J^{(k)}M$ which projects to X in M and which μ-preserves the contact structure is said to be the **μ-prolongation** of order k, or the k-th μ-prolongation, of X. ♣

In order to discuss vector fields in $J^{(k)}M$ which are μ-prolongations of vector fields in M, it will be convenient to agree on a general notation. That is, we write a general vector field in $J^{(k)}M$ in the form

$$Y \ = \ \xi^i \frac{\partial}{\partial x^i} \ + \ \Psi^a_J \frac{\partial}{\partial u^a_J} \ . \tag{20}$$

Theorem 5. *The vector field Y μ-preserves the standard contact structure Θ if and only if its coefficients satisfy the* **vector μ-prolongation formula**

$$\Psi^a_{J,i} \ = \ [\delta^a_b D_i \ + \ (\Lambda_i)^a_b] \ \Psi^b_J \ - \ u^b_{J,k} \, [\delta^a_b D_i \ + \ (\Lambda_i)^a_b] \ \xi^k \ . \tag{21}$$

Proof. This follows easily by a computation analogous to that in the proof of theorem 1. ♢

Theorem 6. *Let $X = \xi^i(\partial/\partial x^i) + \varphi^a(\partial/\partial u^a)$ be a vector field in M. Let $\mu = (\Lambda_i)^a_b \mathrm{d}x^i$ be a $\mathcal{G}$-valued semibasic one-form. Then the coefficients Ψ^a_J of the μ-prolongation Y of X are expressed in terms of the coefficients $\widetilde{\Psi}^a_J$ of the ordinary prolongation of the same vector field X as $\Psi^a_J = \widetilde{\Psi}^a_J + F^a_J$ where the difference terms F^a_J satisfy the recursion relation (with $F^a_0 = 0$)*

$$F^a_{J,i} \ = \ [\delta^a_b D_i \ + \ (\Lambda_i)^a_b] \ F^b_J \ + \ (\Lambda_i)^a_b \, D_J Q^b \ . \tag{22}$$

Proof. Follow the scheme used in the proof of theorem 3. ♢

Similarly to what happens for the λ_i in the scalar case, see remark 3, the (matrix) coefficients Λ_i of the form μ are not completely arbitrary, as they must satisfy some compatibility condition. It is convenient to define the (matrix) operators $\nabla_i := ID_i + \Lambda_i$.

Theorem 7. *The compatibility condition for the matrix coefficients Λ_i of the $\mathcal{G}$-valued form $\mu = (\Lambda_i)^a_b \mathrm{d}x^i$ reads*

$$D_i \, \Lambda_j \ - \ D_j \, \Lambda_i \ + \ [\, \Lambda_i \, , \, \Lambda_j \,] \ = \ 0 \tag{23}$$

for all $i, j = 1, ..., p$. This is equivalent to $[\, \nabla_i \, , \, \nabla_j \,] \ = \ 0$.

Proof. We will use the shorthand notation $(\nabla_i)^a_b = [\delta^a_b D_i + (\Lambda_i)^a_b]$. With this, the vector μ-prolongation formula (21) reads

$$\Psi^a_{J,k} \ = \ (\nabla_k)^a_b \Psi^b_J - u^b_{J,m} (\nabla_k)^a_b \xi^m \ ;$$

applying this twice, we get

$$\Psi^a_{J,k,i} \ = \ \left[(\nabla_i \nabla_k)^a_b \Psi^b_J - u^b_{J,m} (\nabla_i \nabla_k)^a_b \xi^m \right] \ - \ \left[u^b_{J,i,m} (\nabla_k)^a_b + u^b_{J,k,m} (\nabla_i)^a_b \right] \xi^m \ ;$$

note the second square bracket is symmetric in the indices i, k. Thus

$$\left(\Psi^a_{J,k,i} \ - \ \Psi^a_{J,i,k} \right) \ = \ [\nabla_i , \nabla_k]^a_b \Psi^b_J - u^b_{J,m} [\nabla_i , \nabla_k]^a_b \xi^m \ .$$

As for the commutator $[\nabla_i, \nabla_k]$, this is easily computed to be

$$[\nabla_i , \nabla_k] \ = \ D_i \Lambda_k \ - \ D_k \Lambda_i \ + \ [\Lambda_i , \Lambda_k] \ ,$$

i.e. the expression given in the statement, see (23). $\diamondsuit$

Remark 9. One could consider matrices Λ_i belonging to a gauged Lie algebra. By this we mean that $\Lambda_i = \lambda_i^k (x, u^{(1)}) \, L_k$, with $\lambda_i : J^{(1)} M \to \mathbf{R}$ smooth functions and where the L_k ($k = 1, ..., r$) are generators of a (matrix) Lie algebra $\mathcal{G}$, so that $[L_i , L_j] = c^k_{ij} L_k$. In this case the compatibility condition reads $\left[(D_i \lambda^k_j - D_j \lambda^k_i) + c^k_{ab} \lambda^a_i \lambda^b_j \right] \ L_k = 0$; the term in square brackets must vanish for each k. $\odot$

5.2 μ-symmetries and reduction of PDE systems

We define μ-symmetries as in the scalar case; that is, X is a μ-symmetry of a given PDEs system if its μ-prolongation is tangent to the solution manifold of the system. For scalar equations, μ-symmetries can be used to obtain invariant solutions; the same holds for the vector case.

We will assume without further mention that X satisfy the **transversality condition** in the bundle (M, π, B) [10]; see remark 7.

Definition 7'. Let (M, π, B) be a vector bundle over the p-dimensional manifold B, with fiber $\pi^{-1}(x) = U = \mathbf{R}^q$. Let $\Delta = \{ \Delta_1 , ..., \Delta_r \}$ be a system of PDEs of order n for $u^a = u^a (x)$, $a = 1, ..., q$, $x = (x^1 , ..., x^p) \in B$, with solution manifold $S_\Delta \subset J^{(n)} M$. Let X be a vector field in M, and μ a $g\ell(q)$-valued semibasic one-form on M satisfying the compatibility condition (23). Let Y be the μ-prolongation of order n of X. If $Y : S_\Delta \to \mathrm{T} S_\Delta$, we say that X is a **μ-symmetry** of Δ.

In the case of scalar equations, the possibility of using μ-symmetries to perform symmetry reduction relied ultimately on two facts: (i) the space $\mathcal{I}_X$ of X-invariant functions is Y-invariant for Y a μ-prolongations of X; (ii) the standard and the μ-prolongations of X coincide in $\mathcal{I}_X$. This entails that the results valid for reduction of an equation Δ on $\mathcal{I}_X$ under standard symmetries extend to the case of μ-symmetries. The same holds in the case of PDEs systems.

Let us first recall that if $X = \xi^i (\partial / \partial x^i) + \varphi^a (\partial / \partial u^a)$ is a vector field in M, we denote by $Q^a := \varphi^a - u^a_i \xi^i$ its characteristic vector. Then the **X-invariant manifold** in $J^{(n)} M$ is the subset $\mathcal{I}_X \subset J^{(n)} M$ identified by $D_J Q^a = 0$ for all $a = 1, ..., q$ and all multiindices J with $0 \le |J| \le n - 1$.

Theorem 8. *In the hypotheses of theorem 6, let Y be the μ-prolongation of the vector field X. Then Y coincides with the standard prolongation of the same vector field X on $\mathcal{I}_X$.*

Proof. This follows from theorem 6 and the definition of $\mathcal{I}_X$. Indeed, write Ψ^a_J in the form $\Psi^a_J = \widetilde{\Psi}^a_J + F^a_J$, see theorem 6, and suppose that for $|J| = k$ the difference term F^a_J is written as a combination of the $D_J Q^b$, i.e. $F^a_J = (\Gamma^J)^a_b D_J Q^b$. Then from (22) we have

$$F^a_{J,i} \ = \ \delta^a_b [D_i (\Gamma^J)^b_c] (D_J Q^c) + (\Lambda_i)^a_b \left[(\Gamma^J)^b_c (D_J Q^c) + D_J Q^b \right] \ ;$$

this is again a combination of terms of the form $D_J Q^b$. Thus if the F^a_J vanish on $\mathcal{I}_X$ for $|J| = k$, the F^a_J with $|J| \geq k$ also vanish on $\mathcal{I}_X$.

Note that $F^a_i = (\Lambda_i)^a_b Q^b$, so that the condition is satisfied for $|J| = 1$, and the proof of the theorem follows by the recursive computation above. $\diamondsuit$

6 Examples

In the examples below we will consider PDEs in two independent variables, (x,t). In this case we will also write $X = \xi \partial_x + \tau \partial_t + \varphi \partial_u$, and $\mu = \alpha \mathrm{d}x + \beta \mathrm{d}t$.

6.1 μ-symmetries of given equations

In order to determine μ-symmetries of a given PDE Δ of order n, one can proceed in the same way as for ordinary symmetries. That is, consider a generic vector field X acting in M, and its μ-prolongation Y of order n for a generic $\mu = \lambda_i \mathrm{d}x^i$, acting in $J^{(n)}M$. One then applies Y to Δ, and restricts the obtained expression to the solution manifold $S_\Delta \subset J^{(n)}M$. The equation Δ_* resulting by requiring this is zero is the determining equation for μ-symmetries of Δ; this is an equation for ξ, τ, φ and λ_i, and as such is nonlinear.

If we require λ_i are a function on $J^{(k)}M$, all the dependencies on u_J with $|J| > k$ will be explicit, and one obtains a system of determining equations. This system (or the equation Δ_*) should be complemented with the compatibility conditions between the λ_i.

If we determine apriori the form μ, we are left with a system of linear equations for ξ, τ, φ; similarly, if we fix a vector field X and try to find the μ for which it is a μ-symmetry of the given equation Δ, we have a system of quasilinear equations for the λ_i.

The heat equation

Let us first consider the heat equation

$$u_t \ = \ u_{xx} \ ;$$

we will use the ansatz $\mu = \lambda_i \mathrm{d}x^i$ (here $x^1 = x, x^2 = t$),

$$\lambda_i \ = \ D_i \, P(x,y,u) \ ; \tag{24}$$

this guarantees that the compatibility condition (10) is satisfied everywhere (not just on S_Δ).

Proceeding as mentioned above, we obtain the determining equations for μ-symmetries of the heat equation [under the ansatz (24)]; these result to be

$$\begin{array}{l} 2P_u\tau + 2\tau_u = 0 \ , \\ 2P_x\tau + 2\tau_x = 0 \ , \\ P_u^2\xi + P_{uu}\xi + 2P_u\xi_u + \xi_{uu} = 0 \ , \\ P_u^2\tau + P_{uu}\tau + 2P_u\tau_u + \tau_{uu} = 0 \ , \\ -\varphi_{xx} + \varphi_y - 2\varphi_x P_x - \varphi P_x^2 - \varphi P_{xx} + \varphi P_y = 0 \ , \\ P_x^2\tau + P_{xx}\tau - P_y\tau + 2P_x\tau_x + tau_{xx} - \tau_y + 2P_x\xi + 2\xi_x = 0 \ , \\ 2P_uP_x\tau + 2P_{xu}\tau + 2P_x\tau_u + 2P_u\tau_x + 2\tau_{xu} + 2P_u\xi + 2\xi_u = 0 \ , \\ -\varphi_{uu} - 2\varphi_uP_u - \varphi P_u^2 - \varphi P_{uu} + 2P_uP_x\xi + 2P_{xu}\xi + 2P_x\xi_u + 2P_u\xi_x + 2\xi_{xu} = 0 \ , \\ -2\varphi_{xu} - 2\varphi_xP_u - 2\varphi_uP_x - 2\varphi P_uP_x - 2\varphi P_{xu} + P_x^2\xi + P_{xx}\xi - P_y\xi + \\ \quad +2P_x\xi_x + \xi_{xx} - \xi_y = 0 \ . \end{array}$$

After some (lengthy but completely standard) computations, we obtain that the more general solution to these is given by

$$\begin{array}{rcl} \xi(x,t,u) &=& e^{-P} \ [c_1 + c_2 t + (c_3/2)x + (c_4/2)xt] \ , \\ \tau(x,t,u) &=& e^{-P} \ \left[c_5 + c_3 t + (c_4/2)t^2\right] \ , \\ \varphi(x,t,u) &=& e^{-P} \ \left[\zeta(x,t) \ + \ \left(-(c_2/2)x - (c_4/8)(x^2 - 2xt) + c_6\right) u\right] \ , \end{array}$$

where c_i are arbitrary constants, and $\zeta(x,t)$ is an arbitrary function satisfying $\zeta_t = \zeta_{xx}$. Thus, we just obtain the standard symmetries of the heat equation [4, 10, 13], with the factor $\exp[-P(x,y,u)]$; this is no accident, but follows from the ansatz (24), see remark 5 (see also [2]). The characteristic $Q := \varphi - \xi u_x - \tau u_t$ will be the same as for standard symmetries (with a factor e^{-P}), and the symmetry reduced equations will give nothing new.

The Euler equation

Let us consider the Euler equation

$$u_t \ + \ u\, u_x \ = \ 0 \ ;$$

we will write as usual $X = \xi\partial_x + \tau\partial_t + \varphi\partial_u$, and $\mu = \alpha \mathrm{d}x + \beta \mathrm{d}y$.

The condition for X to be a μ-symmetry for the Euler equation is that

$$\begin{array}{c} \varphi u_x + \alpha u\varphi + \beta\varphi + u^2 u_x\alpha\tau + uu_x\beta\tau - uu_x\alpha\xi - u_x\beta\xi + \\ +\varphi_t + uu_x\tau_t - u_x\xi_t + u\varphi_x + u^2 u_x\tau_x - uu_x\xi_x \ = \ 0 \ . \end{array} \tag{25}$$

This should be complemented with the requirement that $D_x\beta = D_t\alpha$ when $u_t + uu_x = 0$. With the ansatz

$$\alpha = \alpha(x,t,u) \ , \ \beta = \beta(x,t,u) \ , \tag{26}$$

(note (24) cannot be verified in this case if α and β do actually depend on u) the dependence of the equations above in u_x is explicit, and (25) splits into two

equations:

$$\begin{aligned}(\alpha u+\beta)\varphi+\varphi_t+u\varphi_x &= 0\ ;\\ \varphi+(\alpha u^2+\beta u)\tau-(\alpha u+\beta)\xi+u\tau_t-\xi_t+u^2\tau_x-u\xi_x &= 0\ .\end{aligned}$$

These are again nonlinear equations for the functions $(\alpha,\beta,\xi,\tau,\varphi)$. A special solution is provided e.g. by

$$\begin{aligned}&\alpha=u\ ,\ \beta=-u^2/2\ ;\\ &\xi=0\ ,\ \tau=[B(u)-A(u)t/u]\ \exp[-(u^2/2)t]\ ,\ \varphi=A(u)\ \exp[-(u^2/2)t]\ .\end{aligned}$$

Note that for this μ, the compatibility conditions $D_t\alpha=D_x\beta$ is satisfied only on the solution manifold S_Δ, see remark 4.

This μ-symmetry corresponds to a nonlocal ordinary symmetry Z of exponential type, see remark 5. We have in facts

$$Z\ =\ e^{\int(u\mathrm{d}x-(u^2/2)\mathrm{d}t)}\ X\ .$$

6.2 Equations with given μ-symmetries

We can also consider the opposite question, i.e. given a vector field X and a form $\mu=\lambda_i\mathrm{d}x^i$ satisfying (10), determine the equations of a given order n which admit X as a μ-symmetry with the given μ.

To solve this problem, we have to consider the μ-prolongation Y of X to $J^{(n)}M$, and solve the characteristic equation for it. In this way we obtain the differential invariants for Y, and any equations which is written in terms of these will admit X as a (strong) μ-symmetry.

Example 1

As a first example, to be dealt with in detail, we will consider μ-prolongations of the scaling vector field

$$X=x\partial_x+2t\partial_t+u\partial_u\ .$$

The invariant coordinates (y,v) and the parametric coordinate σ in $M=\{(x,t,u)\}$ can be chosen as $\sigma=x\ ,\ y=x^2/t\ ,\ v=u/x$; the corresponding inverse change of variables is $x=\sigma\ ,\ t=\sigma^2/y\ ,\ u=\sigma v$.

It follows easily that in the symmetry-adapted coordinates, $X=\sigma\partial_\sigma$; hence the function $v=v(\sigma,y)$ is X-invariant if and only if $v_\sigma=0$, as required by the general method (indeed, by the very definition of symmetry-adapted coordinates).

Applying the procedure described in sect.4, we have

$$u_x=v+2yv_y+\sigma v_\sigma\ ;\ u_t=-(y^2/\sigma)v_y\ .$$

The above can be inverted to give $v_\sigma=(1/x)[u_x+2(t/x)u_t-u/x]=-Q/x^2$, $v_y=-[t^2/x^3]u_t$. Similarly, at second order we get

$$\begin{aligned}&u_{xx}=2v_\sigma+6(y/\sigma)v_y+\sigma v_{\sigma\sigma}+4yv_{\sigma y}+4(y^2/\sigma)v_{yy}\ ,\\ &u_{xt}=-3(y^2/\sigma^2)v_y-(y^2/\sigma)v_{\sigma y}-2(y^3/\sigma^2)v_{yy}\ ,\\ &u_{tt}=2(y^3/\sigma^3)v_y+(y^4/\sigma^3)v_{yy}\ .\end{aligned}$$

We will now consider the simplest nontrivial choice for μ, i.e. $\mu = \lambda \mathrm{d}x$, with λ a real constant.

The second μ-prolongations can be written in the form (20) (with $q = 1$), and general explicit expressions for the coefficients are obtained using either (12) or (14). With our choice for μ, one gets

$$\begin{array}{l} \Psi^x = \lambda\,(u - x u_x - 2t u_t)\ ,\ \Psi^t = -u_t\ , \\ \Psi^{xx} = -u_{xx} - 2\lambda(x u_{xx} + 2t u_{xt}) + \lambda^2 (u - x u_x - 2t u_t)\ , \\ \Psi^{xt} = -2u_{xt} - \lambda(x u_{xt} + 2t u_{tt} + u_t)\ ,\ \Psi^{tt} = -3u_{tt}\ . \end{array}$$

By the method of characteristics, we obtain with standard computations that the invariants of Y in $J^{(2)}M$ are given by:

$$\begin{array}{l} y := (x^2/t)\ ,\quad v := (u/x)\ ; \\ \zeta_1 := x u_t\ ,\quad \zeta_2 := (u/x - 2t u_t/x - u_x) e^{\lambda x} \\ \eta_1 := x t u_{tt}\ ,\quad \eta_2 := \left(x u_t + 2 x t u_{tt} + x^2 u_{xt} \right) e^{\lambda x}\ , \\ \eta_3 := (1/x) \left[(1 - \lambda x)(u - x u_x) + 2\lambda x t u_t + x^2 u_{xx} + 4 x t u_{xt} + 4 t^2 u_{tt} \right] e^{2\lambda x}\ . \end{array}$$

Thus, any equation $\Delta :=\ F\,[y, v, \zeta_1, \zeta_2, \eta_1, \eta_2, \eta_3] = 0$, with F an arbitrary smooth function of its arguments, admits the vector field X given above as a μ-symmetry, with $\mu = \lambda \mathrm{d}x$. Moreover, for $(\partial F/\partial \zeta_2)^2 + (\partial F/\partial \eta_2)^2 + (\partial F/\partial \eta_3)^2 \neq 0$, X is not an ordinary symmetry of Δ.

Finally, it is easily seen by restricting the functions given above to $\mathcal{I}_X$ that the μ-symmetry reduced equation, providing X-invariant solutions to Δ, is given by

$$H[y, v, \zeta_1, \eta_1]\ :=\ F\,[y, v, \zeta_1, 0, \eta_1, 0, 0]\ =\ 0\ .$$

Let us discuss a completely concrete example. Consider the equation $\Delta := \eta_1 - \zeta_2 = 0$. This is written as

$$x t\, u_{tt}\ +\ (u_x + (2t/x) u_t - u/x)\ e^{\lambda x}\ =\ 0$$

in the original coordinates; in the adapted ones it reads

$$y^3 v_{yy} + 2y^2 v_y + \sigma e^{\lambda \sigma} v_\sigma\ =\ 0\ .$$

The corresponding reduced equation is $y^2[y v_{yy} + 2v_y] = 0$; the general solution to this is $v(y) = c_1 + c_2/y$, where c_i are real constants. Going back to the original coordinates, the corresponding solutions are $u(x,t) = (c_1 x^2 + c_2 t)/x$.

Example 2.

Consider the same X as above, with $\mu = -(1/t)\mathrm{d}t$. In this case the Y-invariant functions are spanned by

$$\begin{array}{l} \zeta_1 = u_x\ ;\quad \zeta_2 = u_t/x - (u/x - u_x)/(2t)\ ; \\ \eta_1 = x u_{xx}\ ,\quad \eta_2 = u_{xt} + x u_{xx}/(2t)\ , \\ \eta_3 = (x u_{xx} - 3u_x)/(4t^2) + u_{xt}/t + u_{tt}/x - u_t/(xt) + 3u/(4xt^2)\ . \end{array}$$

Any equation given by $F[y,v,\zeta_1,\zeta_2,\eta_1,\eta_2,\eta_3]=0$ will admit X as a (strong) μ-symmetry, and if $(\partial F/\partial\zeta_2)^2+(\partial F/\partial\eta_2)^2+(\partial F/\partial\eta_3)^2\neq 0$, X is not an ordinary symmetry.

The restriction of the invariant functions to $\mathcal{I}_X$ yields $\zeta_1=u_x$, $\zeta_2=0$, and $\eta_1=xu_{xx}$, $\eta_2=\eta_3=0$; hence the reduced equation will be simply

$$H[y,v,\zeta_1,\eta_1] \ := \ F[y,v;\zeta_1,0;\eta_1,0,0] \ = \ 0 \ .$$

Example 3.

Any equation of the form $F[y,v,\zeta_1,\zeta_2,\eta_1,\eta_2,\eta_3] = 0$, where F is a smooth function of its arguments and we have defined

$$\begin{array}{l} y=\sqrt{x^2+t^2} \ , \quad v=u \ , \\ \zeta_1=u_x/x \ , \quad \zeta_2=u_t-(t/x)u_x \ , \\ \eta_1=(y^2/x^3)(xu_{xx}-u_x) \ , \quad \eta_2=(y/x^3)(xtu_{xx}-x^2u_{xt}-tu_x) \ , \\ \eta_3=u_{tt}+(t/x^3)(xtu_{xx}-2x^2u_{xt}-tu_x) \ , \end{array}$$

admits the rotation vector field

$$X \ = \ x\partial_t-t\partial_x$$

as a μ-symmetry, with $\mu=-(1/x)\mathrm{d}x$.

The functions y,v provide invariants in M, and we can select $\sigma=\mathrm{arctg}(t/x)$; the inverse change of coordinates is given by $x=y\cos(\sigma)$, $t=y\sin(\sigma)$, $u=v$. The vector field X is then expressed as $X=\partial_\sigma$

The invariant subset $\mathcal{I}_X$ is in this case identified by $u_t=(t/x)u_x$, $u_{xt}=(xu_{xx}-u_x)(t/x^2)$, $u_{tt}=[xt^2u_{xx}+(x^2-t^2)u_x]/x^3$. The restriction of the invariant functions to $\mathcal{I}_X$ yields $\zeta_1=u_x/x$, $\zeta_2=0$, $\eta_1=(r^2/x^3)(xu_{xx}-u_x)$, $\eta_2=0$, $\eta_3=u_x/x$ (note $\eta_3=\zeta_1$); hence the reduced equation will be simply

$$H[y,v,\zeta_1,\eta_1] \ := \ F[y,v;\zeta_1,0;\eta_1,0,\zeta_1] \ = \ 0 \ .$$

Example 4.

In previous examples the functions λ and ν in $\mu=\lambda\mathrm{d}x+\nu\mathrm{d}t$ were always depending only on x and t; in this last example they will depend on first order derivatives of the u.

Any equation of the form $F(y,v;\zeta_1,\zeta_2;\eta_1,\eta_2,\eta_3)=0$ with F a smooth function of its arguments, which are

$$\begin{array}{l} y \ = \ t \ , \quad v \ = \ u/x \ ; \\ \zeta_1=u+\log[1-u/(xu_x)] \ , \quad \zeta_2=-(uu_t)/(x^2u_x) \ ; \\ \eta_1 \ = \ -e^{2u}\left[u/(x^2u_x^3)\right]\left[u^2(u_x^2-u_{xx})+xu_x^3-(1+xu_x)uu_x^2\right] \ , \\ \eta_2 \ = \ e^u\left[u^2/(x^3u_x^3)\right]\left[xu_xu_{xt}-(u_x+xu_{xx})u_t\right] \ , \\ \eta_3 \ = \ \left[u/(x^3u_x^3)\right]\left[xu_x^2u_{tt}-2xu_xu_tu_{xt}+(2u_x+xu_{xx})u_t^2\right] \ , \end{array}$$

admits the scaling vector field

$$X \ = x\,\partial_x \ + \ u\,\partial_u$$

as a (strong) μ-symmetry, with $\mu = u_x \mathrm{d}x + u_t \mathrm{d}t$.

The invariant set $\mathcal{I}_X$ is identified by $u_x = u/x$, $u_{xt} = u_t/x$, $u_{xx} = 0$. Restriction of the invariant functions to $\mathcal{I}_X$ yields $\zeta_1 = \zeta_1^0 := u/x = v^2$, $\zeta_2 = \zeta_2^0 := -u_t/x$, $\eta_1 = \eta_2 = 0$, $\eta_3 = \eta_3^0 := u_{tt}/x$. Hence the reduced equation will be simply

$$H[y, v, \zeta_2^0, \eta_3^0] \ := \ F[y, v; v, \zeta_2^0; 0, 0, \eta_3^0] \ = \ 0 \ .$$

6.3 Systems of PDEs

Finally, we consider systems of PDEs with a given μ-symmetry, and the corresponding reduction. We will denote independent and dependent variables as (x, y) and (u, v) respectively.

Example 1

Let us consider the scaling vector field

$$X \ = \ x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + u\frac{\partial}{\partial u} + 2v\frac{\partial}{\partial v}$$

and the form $\mu \ = \ \lambda I \mathrm{d}x$ with λ a real constant; this corresponds to matrices Λ_i given by $\Lambda_{(x)} = \lambda I$ and $\Lambda_{(y)} = 0$.

By applying the vector μ-prolongation formula (21), or using theorem 6 and (22), we determine the second μ-prolongation Y of X. We can then solve the characteristic equation for the flow of Y in $J^{(2)}M$, and determine a basis of Y-invariant functions. Such a basis is provided by the following set of functions:

$$\begin{array}{l}
\rho = x^2/y \ , \ w_1 := u/x \ , \ w_2 := v/x^2 \ ; \\
\zeta_1 := y u_y/x \ , \ \zeta_2 := (u_x + 2y u_y/x - u/x) e^{\lambda x} \ , \\
\zeta_3 := v_y \ , \ \zeta_4 := (v_x/x - 2v/x^2 + 2y v_y/x^2) e^{\lambda x} \ ; \\
\eta_1 := x^2 v_{yy} \ , \ \eta_2 := x^3 u_{yy} \ , \\
\eta_3 := (x v_{xy} + 2y v_{yy}) e^{\lambda x} \ , \\
\eta_4 := (x^2 u_{xy} + x u_y + 2xy u_{yy}) e^{\lambda x} \ , \\
\eta_5 := [v_{xx} + 4v/x^2 - 3v_x/y + 4y u_{xy} - 4y v_y/x^2 + 4y u_y/x + \\
\quad -4y^2 v_{yy}/x^2 + 8y^2 u_{yy}/x + \lambda(v_x - 2v/x + 2y v_y/x)] e^{2\lambda x} \ , \\
\eta_6 := [-u_x + u/x + x u_{xx} + 4y u_{xy} + 4y^2 u_{yy}/x + \\
\quad +\lambda(-u + x u_x + 2y u_y)] e^{2\lambda x} \ .
\end{array} \tag{27}$$

Any (system of) second order equation of the form

$$F^i \left[y, w_1, w_2; \zeta_1, ..., \zeta_4; \eta_1, ..., \eta_6 \right] = 0 \tag{28}$$

with F^i $(i = 1, ..., n)$ a smooth function of its arguments, admits X as a (strong) μ-symmetry.

[2] We stress that this expression for ζ_1 is not obtained by a direct substitution: indeed now $Q = 0$ means $u_x = u/x$; the general expression for ζ_1 given above becomes singular, but the expression for Ψ^x guarantees that u_x is constant, and actually equal to $u/x = v$ on $Q = 0$.

In order to consider the μ-symmetry reduced equation, it suffices to consider the restriction of the functions ζ_i, η_j on $\mathcal{I}_X$ (note that the μ-prolongation and the ordinary one coincide on $\mathcal{I}_X$, see lemma 5.) The manifold $\mathcal{I}_X$ is identified by $Q = D_x Q = D_y Q = 0$; in the present case these mean

$$\begin{array}{lll} u_y = (u - x u_x)/(2y) \ , & u_{xy} = -(x u_x x)/(2y) \ , & u_{yy} = -(u_y + x u_{xy})/(2y) \ ; \\ v_y = (2v - x v_x)/(2y) \ , & v_{xy} = (v_x - x v_{xx})/(2y) \ , & v_{yy} = -(x v_{xy})/(2y) \ . \end{array}$$

Substituting these into (27) above, we obtain the expressions for the reduction of first and second order Y-invariants restricted to $\mathcal{I}_X$, which are

$$\begin{array}{l} \zeta_1^0 = (u - u_x\, x)/(2\, x) \ , \ \zeta_2^0 = 0 \ , \\ \zeta_3^0 = (2v - x v_x)/(2y) \ , \ \zeta_4^0 = 0 \ ; \\ \eta_1^0 = (x^3\, (-v_x + x v_{xx}))/(4y^2) \ , \\ \eta_2^0 = (x^3\, (-u + x\, (u_x + x u_{xx})))/(4y^2) \ , \\ \eta_3^0 = 0 \ , \ \eta_4^0 = 0 \ , \ \eta_5^0 = 0 \ , \ \eta_6^0 = 0 \ . \end{array}$$

Thus, the X-invariant solutions of (28) are obtained as solution of the reduced system of equations

$$H^i[y, w_1, w_2; \zeta_1^0, \zeta_3^0, \eta_1^0, \eta_2^0] \ := \ F^i\left[y, w_1, w_2; \zeta_1^0, 0, \zeta_3^0, 0; \eta_1^0, \eta_2^0, 0, 0, 0, 0\right] = 0 \ .$$

Example 2

Consider next the elementary vector field $X = x\partial_x$, and the form $\mu = \Lambda_{(x)} \mathrm{d}x + \Lambda_{(y)} \mathrm{d}y$ corresponding to matrices

$$\Lambda_x \ = \ \begin{pmatrix} 0 & -y \\ 0 & 0 \end{pmatrix} \ , \ \Lambda_y = \frac{1}{y^2} \begin{pmatrix} -xy & (x^2 - x)y^2 \\ 1 & xy \end{pmatrix} \ .$$

In this case, a basis for Y-invariant functions on $J^{(2)}M$ is provided by

$$\begin{array}{l} \rho = y \ , \quad w_1 = u \ , \quad w_2 = v \ ; \\ \zeta_1 = x u_x - x^2 y v_x \ , \quad \zeta_2 = x v_x \ , \\ \zeta_3 = (1/y)[y u_y - x^2 y v_x - x(x u_x - x^2 y v_x)] \ , \\ \zeta_4 = v_y - (1/y^2)\, (x u_x - x^2 y v_x)\, \log(x) \ ; \\ \eta_1 = x^2 u_{xx} - 2x^2 y v_x - 2x^3 y v_{xx} \ , \quad \eta_2 = x^2 v_{xx} \ , \\ \eta_3 = x[u_{xy} - x(2v_x + x v_{xx} + y v_{xy})] \ , \\ \eta_4 = -(1/y^2)\{xy(x v_x + x^2 v_{xx} - y v_{xy}) + x[u_x + x(u_{xx} - 3y v_x - 2xy v_{xx})] \log(x)\} \ , \\ \eta_5 = (1/y^2)[4x^2 u_x + 2x^3 u_{xx} + y(-2x^2 u_{xy} - 4x^3 v_x - 2x^4 v_{xx} + \\ \quad + y u_{yy} - 2x^2 y v_{xy} + 2x^3 y v_{xy})] \ , \\ \eta_6 = (1/y^3)\{-2x^2 y v_x - 2x^3 y v_{xx} + y^3 v_{yy} + \\ \quad + 2x[u_x + y(x v_x + x^2 v_{xx} + xy v_{xy} - u_{xy})] \log(x)\} \ . \end{array}$$

Any system of second order equation of the form

$$F^i\left[y, w_1, w_2; \zeta_1, ..., \zeta_4; \eta_1, ..., \eta_6\right] = 0 \ ,$$

F^i a smooth function of its arguments, admits X as a (strong) μ-symmetry.

The system identifying $\mathcal{I}_X$ is now given by

$$
\begin{array}{lll}
x u_x = 0, & u_x + x u_{xx} = 0, & x u_{xy} = 0, \\
x v_x = 0, & v_x + x v_{xx} = 0, & x v_{xy} = 0 \ ;
\end{array}
$$

$\mathcal{I}_X$ is the linear space on which $u_x = u_{xx} = u_{xy} = v_x = v_{xx} = v_{xy} = 0$. Restriction of the invariant functions given above to this space yields

$$
\begin{array}{l}
\zeta_1^0 = \zeta_2^0 = 0 \ , \ \zeta_3^0 = u_y \ , \ \zeta_4^0 = v_y \ ; \\
\eta_1^0 = \eta_2^0 = \eta_3^0 = \eta_4^0 = 0 \ , \ \eta_5^0 = u_{yy} \ , \ \eta_6^0 = v_{yy} \ .
\end{array}
$$

The reduced system is therefore

$$
H^i[y,u,v;u_y,v_y,u_{yy},v_{yy}] \ = \ F^i[y,u,v;0,0,u_y,v_y;0,0,0,0,u_{yy},v_{yy}] \ = \ 0 \ .
$$

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